

DIRECT IMAGES OF SEMI-MEROMORPHIC CURRENTS

MATS ANDERSSON & ELIZABETH WULCAN

ABSTRACT. We introduce a calculus for direct images of semi-meromorphic currents. We also provide results on the behaviour of these currents under the action of holomorphic differential operators and interior multiplication by holomorphic vector fields.

1. INTRODUCTION

Let f be a generically nonvanishing holomorphic function on a reduced analytic space X of pure dimension n . It was proved by Herrera and Lieberman that one can define the principal value current

$$(1.1) \quad \left[\frac{1}{f} \right] . \xi := \lim_{\epsilon \rightarrow 0} \int_{|f|^2 > \epsilon} \frac{\xi}{f},$$

for test forms ξ . It follows that $\bar{\partial}[1/f]$ is a current with support on the zero set $Z(f)$ of f ; such a current is called a residue current. Coleff and Herrera, [15], introduced products of principal value and residue currents, like $[1/f_1] \cdots [1/f_r]$, $\bar{\partial}[1/f_1] \wedge \cdots \wedge \bar{\partial}[1/f_r]$, $[1/f_1] \wedge \bar{\partial}[1/f_2]$ etc. The product of principal value currents is commutative, but when there are residue factors, like $\bar{\partial}[1/f]$, present these products are not (anti-)commutative in general. In the literature there are various generalizations and related currents, for instance the abstract Coleff-Herrera currents in [13], the Bochner-Martinelli type residue currents introduced in [23], and generalizations in, e.g., [1], [3], and [9].

Following [7] we say that a current a is *almost semi-meromorphic*, $a \in ASM(X)$, if it is the direct image under a modification of a semi-meromorphic current, i.e., of the form $\omega \wedge [1/f]$, where f is holomorphic and ω is smooth. Almost semi-meromorphic currents in many ways generalize principal value currents; for example, the form an (anti-)commutative algebra, see Section 4. The class $ASM(X)$ is closed under $\bar{\partial}$, see Proposition 4.10. Taking $\bar{\partial}$ of $a \in ASM(X)$, however, yields an almost semi-meromorphic current plus a residue current supported on the Zariski closure of the set where a is nonsmooth. The residue currents mentioned above can all be constructed as the residue currents of certain almost semi-meromorphic currents.

In order to obtain a coherent approach to questions about residue and principal value currents we introduced in [10] the sheaf \mathcal{PM}^X of *pseudomeromorphic currents* on X as direct images of, products of semi-meromorphic and $\bar{\partial}$ of semi-meromorphic currents; see Section 2 below for the precise definition. In particular, almost semi-meromorphic currents are pseudomeromorphic. Pseudomeromorphic currents have a geometric nature, similar to positive closed (or normal) currents. For example, the *dimension principle* states that if the pseudomeromorphic current τ has bidegree

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$(*, p)$ and support on a variety of codimension larger than p , then τ must vanish. Moreover one can form the restriction $\mathbf{1}_W \tau$ of the pseudomeromorphic current τ to the analytic, (or constructible) subset $W \subset X$, such that

$$(1.2) \quad \mathbf{1}_V \mathbf{1}_W \tau = \mathbf{1}_{V \cap W} \tau.$$

See Section 3. The notion of pseudomeromorphic currents plays a decisive role in for instance in [11], [8], [18], [19], [7], and [25].

In general we cannot give any reasonable meaning to the product of two pseudomeromorphic currents. One of the main results in this paper, Theorem 4.6 asserts that given an almost semi-meromorphic current a and a pseudomeromorphic current τ on X one can form a pseudomeromorphic current T that coincides with $a \wedge \tau$ outside the Zariski closure V of the set where a is non-smooth, and such that T has no mass at V , i.e. $\mathbf{1}_V T = 0$. For $a = [1/f]$ this was proved in [10] and various other special cases of Theorem 4.6 have appeared in the papers mentioned above.

In case X is smooth, we prove in Sections 5 and 6 that $ASM(X)$ and \mathcal{PM}^X , respectively, are closed under the action of holomorphic differential operators and interior multiplication by holomorphic vector fields.

2. PSEUDOMEROMORPHIC CURRENTS

In one complex variable s one can define the principal value current $[1/s^m]$ for instance as the value

$$\left[\frac{1}{s^m} \right] = \left. \frac{|s|^{2\lambda}}{s^m} \right|_{\lambda=0}$$

of the current-valued analytic continuation of $\lambda \mapsto |s|^{2\lambda}/s^m$, a priori defined for $\operatorname{Re} \lambda \gg 0$, see, e.g., [1, Lemma 2.1]. We have the relations

$$(2.1) \quad \frac{\partial}{\partial s} \left[\frac{1}{s^m} \right] = -m \left[\frac{1}{s^{m+1}} \right], \quad s \left[\frac{1}{s^{m+1}} \right] = \left[\frac{1}{s^m} \right].$$

It is also well-known that

$$(2.2) \quad \bar{\partial} \left[\frac{1}{s^{m+1}} \right] \cdot \xi ds = \frac{1}{m!} \frac{\partial^m}{\partial s^m} (0).$$

for test functions ξ ; in particular, $\bar{\partial}[1/s^{m+1}]$ has support at $\{s = 0\}$. Thus

$$(2.3) \quad \bar{s} \bar{\partial} \left[\frac{1}{s^{m+1}} \right] = 0, \quad d\bar{s} \wedge \bar{\partial} \left[\frac{1}{s^{m+1}} \right] = 0.$$

We say that a function χ on the real line is a *smooth approximand of the characteristic function* $\chi_{[1, \infty)}$ of the interval $[1, \infty)$, and write

$$\chi \sim \chi_{[1, \infty)},$$

if χ is smooth, equal to 0 in a neighborhood of 0 and 1 in a neighborhood of ∞ . It is well-known that $[1/s^m] = \lim \chi(|s|^2/\epsilon)(1/s^m)$.

Let t_j be coordinates in an open set $\mathcal{U} \subset \mathbb{C}^n$ and let α be a smooth form with compact support in \mathcal{U} . Then

$$(2.4) \quad \tau = \alpha \wedge \left[\frac{1}{t_1^{m_1}} \right] \cdots \left[\frac{1}{t_k^{m_k}} \right] \bar{\partial} \left[\frac{1}{t_{k+1}^{m_{k+1}}} \right] \wedge \cdots \wedge \bar{\partial} \left[\frac{1}{t_r^{m_r}} \right]$$

is a well-defined current, since it is the tensor product of one-variable currents (times α). We say that τ is an *elementary pseudomeromorphic current*, and we refer to $[1/t_j^{m_j}]$ and $\bar{\partial}[1/t_\ell^{m_\ell}]$ as its *principal value factors* and *residue factors*, respectively.

It is clear that (2.4) is commuting in the principal value factors and anti-commuting in the residue factors. We say the the affine set $\{t_{k+1} = \dots = t_r = 0\}$ is the *elementary support* of τ . Clearly the support of τ is contained in the intersection of the elementary support and the support of α .

Remark 2.1. In view of (2.1), note that $\partial\tau$ is an elementary current, whose elementary support either equals the elementary support of τ or is empty. Also $\bar{\partial}\tau$ is a finite sum of elementary currents, whose elementary supports are either as above or codimension one subvarieties of the elementary support of τ , cf. (2.1), (2.2). \square

Let X be a reduced complex space of dimension n . Fix a point $x \in X$. We say that a germ μ of a current at x is *pseudomeromorphic* at x , $\mu \in \mathcal{PM}_x$, if it is a finite sum of currents of the form $\pi_*\tau = \pi_*^1 \dots \pi_*^m \tau$, where \mathcal{U} is a neighborhood of x ,

$$(2.5) \quad \mathcal{U}_m \xrightarrow{\pi^m} \dots \xrightarrow{\pi^2} \mathcal{U}_1 \xrightarrow{\pi^1} \mathcal{U}_0 = \mathcal{U},$$

each $\pi^j: \mathcal{U}_j \rightarrow \mathcal{U}_{j-1}$ is either a modification, a simple projection $\mathcal{U}_{j-1} \times Z \rightarrow \mathcal{U}_{j-1}$, or an open inclusion (i.e., \mathcal{U}_j is an open subset of \mathcal{U}_{j-1}), and τ is elementary on $\mathcal{U}_m \subset \mathbb{C}^n$.

By definition the union $\mathcal{PM} = \mathcal{PM}^X = \cup_x \mathcal{PM}_x$ is an open subset of the sheaf $\mathcal{C} = \mathcal{C}^X$ and hence it is a subsheaf, which we call the sheaf of *pseudomeromorphic currents*¹. of \mathcal{C} . A section μ of \mathcal{PM} over an open set $\mathcal{V} \subset X$, $\mu \in \mathcal{PM}(\mathcal{V})$, is then a locally finite sum

$$(2.6) \quad \mu = \sum (\pi_\ell)_* \tau_\ell,$$

where each π_ℓ is a composition of mappings as in (2.5) (with $\mathcal{U} \subset \mathcal{V}$) and τ_ℓ is elementary. For simplicity we will always suppress the subscript ℓ in π_ℓ . If ξ is a smooth form, then

$$\xi \wedge \pi_* \tau = \pi_* (\pi^* \xi \wedge \tau).$$

Thus \mathcal{PM} is closed under exterior multiplication by smooth forms. Since $\bar{\partial}$ and ∂ commute with push-forwards it follows that \mathcal{PM} is closed under $\bar{\partial}$ and ∂ , cf. Remark 2.1.

Remark 2.2. One may assume that each τ_ℓ in (2.6) has at most one residue factor. Indeed, in [23], see also [4], it is shown that the Coleff-Herrera product $\bar{\partial}[1/t_{k+1}^{m_{k+1}}] \wedge \dots \wedge \bar{\partial}[1/t_r^{m_r}]$ equals the Bochner-Martinelli residue current of $t_{k+1}^{m_{k+1}}, \dots, t_r^{m_r}$, which, see, e.g., [1], is the direct image of a current of the form $\bar{\partial}(\alpha \wedge [1/f])$, where f is a monomial. It follows that (2.4) is a finite sum of elementary currents with at most one residue factor. \square

Proposition 2.3. *Assume that $\mu \in \mathcal{PM}$ has support on the subvariety $V \subset X$.*

- (i) *If the holomorphic function h vanishes on V , then $\bar{h}\mu = 0$ and $d\bar{h}\mu = 0$.*
- (ii) *If μ has bidegree $(*, p)$ and $\text{codim } V > p$, then $\mu = 0$.*

This proposition is from [10]; for the adaption to nonsmooth X , see [7, Proposition 2.3]. Part (i) implies that the current τ only involves holomorphic derivatives of test forms. We refer to part (ii) as the *dimension principle*. We will also need, [5, Proposition 1.2],

¹The definition here is from [7] and it is in turn a slight modification of the definition introduced in [10].

Proposition 2.4. *If $p: X' \rightarrow X$ is a modification, then $p_*: \mathcal{PM}(X') \rightarrow \mathcal{PM}(X)$ is surjective.*

3. RESTRICTIONS OF PSEUDOMEROMORPHIC CURRENTS

Assume that μ is pseudomeromorphic on X and that $V \subset X$ is a subvariety. It was proved in [10], see also [7], that the natural restriction of μ to the open set $X \setminus V$ has a natural pseudomeromorphic extension $\mathbf{1}_{X \setminus V} \mu$ to X . In [10] it was obtained as the value

$$(3.1) \quad \mathbf{1}_{X \setminus V} \mu := |f|^{2\lambda} \mu|_{\lambda=0}$$

at $\lambda = 0$ of the analytic continuation of the current valued function $\lambda \mapsto |f|^{2\lambda} \mu$, where f is any holomorphic tuple such that $Z(f) = V$. It follows that

$$\mathbf{1}_V \mu := \mu - \mathbf{1}_{X \setminus V} \mu$$

has support on V . It is proved in [10] that this operation extends to all constructible sets and that (1.2) holds. If α is a smooth form, then

$$(3.2) \quad \mathbf{1}_V(\alpha \wedge \mu) = \alpha \wedge \mathbf{1}_V \mu.$$

Moreover, if $\pi: X' \rightarrow X$ is a modification or simple projection (or open inclusion) and $\mu = \pi_* \mu'$, then

$$(3.3) \quad \mathbf{1}_V \mu = \pi_*(\mathbf{1}_{\pi^{-1}V} \mu').$$

In this paper it is convenient to express $\mathbf{1}_{X \setminus V} \mu$ as a limit of currents that are pseudomeromorphic themselves.

Lemma 3.1. *Let V be the germ of a subvariety at $x \in X$, let f be a holomorphic tuple whose common zero set is precisely V , let v be a smooth and nonvanishing function, and let $\chi \sim \chi_{[1, \infty)}$. For each germ of a pseudomeromorphic current μ at x we have*

$$(3.4) \quad \mathbf{1}_{X \setminus V} \mu = \lim_{\delta \rightarrow 0} \chi(|f|^2 v / \delta) \mu.$$

Because of the factor v , the lemma holds just as well for a holomorphic section f of a Hermitian vector bundle.

In case V is a hypersurface and f is one single holomorphic function, or section of a line bundle, the lemma follows directly from Lemma 6 in [20] by just taking $T = f\mu$. We will reduce the general case to this lemma. The proof of this lemma relies on the proof of Theorem 1.1 in [20], which is quite involved. For a more direct proof of Lemma 3.1 see [6].

Proof. Let $\pi: X' \rightarrow X$ be a smooth modification such that $\pi^* f = f^0 f'$, where f^0 is a holomorphic section of a Hermitian line bundle $L \rightarrow X'$ and f' is a nonvanishing tuple of sections of L^{-1} . In view of Proposition 2.4 we can assume that $\mu = \pi_* \mu'$. Then

$$|\pi^* f|^2 \pi^* v = |f^0|^2 |f'|^2 \pi^* v,$$

and from Lemma 6 in [20] we thus have that

$$\lim_{\epsilon \rightarrow 0} \chi(|\pi^* f|^2 \pi^* v / \epsilon) \mu' = \mathbf{1}_{X' \setminus \pi^{-1}V} \mu'$$

In view of (3.3) we get (3.4). □

Remark 3.2. Lemma 3.1 holds even if $\chi = \chi_{[1,\infty)}$. However, in general it is not at all obvious what $\chi(|f|^2 v/\epsilon)\tau$ means. Let χ^δ be smooth approximands such that $\chi^\delta \rightarrow \chi_{[1,\infty)}$. It follows from the proof of Lemma 6 in [20] that for small enough ϵ , depending on τ , f and v , the limit $\lim_{\delta \rightarrow 0} \chi^\delta(|f|^2 v/\epsilon)\tau$ exists and is independent of the choice of χ^δ ; thus we can take it as the definition of $\chi(|f|^2 v/\epsilon)\tau$. In fact, it turns out that $\chi(|f|^2 v/\epsilon)\tau$ is a tensor product of currents after a suitable change of real coordinates. In particular we get the natural meaning in a case like (1.1). \square

We will need the following useful observation.

Lemma 3.3. *If μ has the form (2.6) then*

$$\mathbf{1}_V \mu = \sum_{\text{supp } \tau_\ell \subset \pi^{-1}V} \pi_* \tau_\ell.$$

Proof. In view of (3.3) we have that

$$\mathbf{1}_V \mu = \sum_{\ell} \pi_* (\mathbf{1}_{\pi^{-1}V} \tau_\ell).$$

If $\text{supp } \tau_\ell \subset \pi^{-1}V$, then clearly $\mathbf{1}_{\pi^{-1}V} \tau_\ell = \tau_\ell$. We now claim that if $\text{supp } \tau_\ell$ is not contained in $\pi^{-1}V$, then $\mathbf{1}_{\pi^{-1}V} \tau_\ell = 0$. If $\text{supp } \tau_\ell \not\subset \pi^{-1}V$, the elementary support H of τ_ℓ is also not contained in $\pi^{-1}V$. Assume that H has codimension q . Then τ_ℓ is of the form $\tau_\ell = \alpha \wedge \tau'$, where α is smooth and τ' is elementary of bidegree $(0, q)$. Since H is a linear subspace it is irreducible, and therefore $\pi^{-1}V \cap H$ has codimension at least $q + 1$. It follows from (3.2) that

$$\mathbf{1}_{\pi^{-1}V} \tau_\ell = \alpha \wedge \mathbf{1}_{\pi^{-1}V} \tau'$$

and $\mathbf{1}_{\pi^{-1}V} \tau'$ must vanish in view of the dimension principle. Now the lemma follows. \square

We now consider another fundamental operation on \mathcal{PM} introduced in [10].

Proposition 3.4 ([10]). *Given a holomorphic function h and a pseudomeromorphic current μ there is a pseudomeromorphic current T such that $T = (1/h)\mu$ in the open set where $h \neq 0$ and $\mathbf{1}_{\{h=0\}}T = 0$.*

Clearly this current T must be unique and we denote it by $[1/h]\tau$.

Remark 3.5. Notice that $h[1/h]\tau = \mathbf{1}_{\{h \neq 0\}}\tau$; in particular, $h[1/h]\tau \neq \tau$ in general. For example, $z(1/z)\bar{\partial}[1/z] = 0$. \square

In [10] the current $[1/h]\tau$ was defined as

$$\left. \frac{|h|^{2\lambda}}{h} \tau \right|_{\lambda=0}.$$

Since $\mathbf{1}_{\{h \neq 0\}}[1/h]\tau = (1/h)\tau$ in $\{h \neq 0\}$ it follows from (3.4) that

$$(3.5) \quad \left[\frac{1}{h} \right] \tau = \lim_{\epsilon \rightarrow 0} \chi(|h|^2 v/\epsilon) \frac{1}{h} \tau.^3$$

Here h may just as well be a holomorphic section of a line bundle.

²We have not excluded the possibility that h vanishes identically on some (or all) irreducible components of X .

³In fact, this is (part of) Lemma 6 in [20].

Let Z be a pure-dimensional subvariety of X . We say that a pseudomeromorphic current μ on X with support on Z has the *standard extension property*, SEP, on Z if $\mathbf{1}_V \mu = 0$ for each $V \subset Z$ of positive codimension in Z . We let \mathcal{W}_Z denote the subsheaf of \mathcal{PM} of currents with the SEP on Z . Instead of \mathcal{W}_X we just write \mathcal{W} .

4. ALMOST SEMI-MEROMORPHIC CURRENTS

We say that a current on X is *semi-meromorphic* if it is a principal value current of the form $\omega \wedge [1/f]$, where ω is a smooth form and f is a holomorphic function or section of a line bundle that is generically nonvanishing on each component of X . For simplicity we mostly suppress the brackets $[\]$ indicating principal value, and just write $(1/f)\tau$ in the sequel.

Let X be a pure-dimensional analytic space. We say that a current a is *almost semi-meromorphic* in X , $a \in ASM(X)$, if there is a modification $\pi: X' \rightarrow X$ such that

$$(4.1) \quad a = \pi_*(\omega/f),$$

where f is a holomorphic section of a line bundle $L \rightarrow X'$, not vanishing identically on any irreducible component of X' , and ω is a smooth form with values in L . We say that a is *almost smooth* in X if one can choose f nonvanishing. We can assume that X' is smooth because otherwise we take a smooth modification $\pi': X'' \rightarrow X'$ and consider the pullbacks of f and ω to X'' . If nothing else is said we always tacitly assume that X' is smooth.

Assume that $a \in ASM(X)$ and that V has positive codimension in X . Then $\pi^{-1}V$ has positive codimension in X' and we have that $\mathbf{1}_V a = \pi_*(\mathbf{1}_{\pi^{-1}V}(\omega/f)) = \pi_*(\omega \mathbf{1}_{\pi^{-1}V}(1/f)) = 0$. Thus $ASM(X)$ is contained in $\mathcal{W}(X)$.

Remark 4.1. One can of course introduce a notion of locally almost semi-meromorphic currents and consider the associated sheaf. However, we will have no immediate need for this notion. \square

Example 4.2. Assume that $X = \{zw = 0\} \subset \mathbb{C}^2$. Let $a: X \rightarrow \mathbb{C}$ be 1 on $X \setminus \{z = 0\}$ and 0 otherwise. Then a is almost smooth. Indeed the normalization $\nu: \tilde{X} \rightarrow X$ consists of two disjoint components and $a = \nu_* \tilde{a}$, where $\tilde{a} = 0$ on one of the components and $\tilde{a} = 1$ on the other component. \square

Given a modification $\pi: X' \rightarrow X$, let $\text{sing}(\pi) \subset X'$ be the (analytic) set where π is not a biholomorphism. By the definition of a modification it has positive codimension. Let $Z \subset X'$ be the zero set of f . By assumption also Z has positive codimension. Notice that $a \in ASM(X)$ is smooth outside $\pi(Z \cup \text{sing}(\pi))$ which has positive codimension in X . We let $ZSS(a)$ denote the smallest Zariski-closed set V such that a is smooth outside V , and we call it the *Zariski-singular support* of a .

Example 4.3. Assume that $a \in ASM(X)$ is almost smooth. Then $a = \pi_* \omega$, where ω is smooth, and thus $ZSS(a) \subset \pi(\text{sing}(\pi))$. However, this inclusion may be strict; for example if a is smooth, then $ZSS(a)$ is empty. Notice that if a is smooth, then $\omega = \pi^* a$ outside $\text{sing}(\pi)$. Since both sides are smooth across $\text{sing}(\pi)$, by continuity this equality must hold everywhere in X' . \square

Given two modifications $X_1 \rightarrow X$ and $X_2 \rightarrow X$ there is a modification $\pi: X' \rightarrow X$ that factorizes over both X_1 and X_2 , i.e., we have $X' \rightarrow X_j \rightarrow X$ for $j = 1, 2$.

Therefore, given $a_1, a_2 \in ASM(X)$ we can assume that $a_j = \pi_*(\omega_j/f_j)$, $j = 1, 2$. It follows that

$$a_1 + a_2 = \pi_*\left(\frac{\omega_1}{f_1} + \frac{\omega_2}{f_2}\right) = \pi_*\frac{f_2\omega_1 + f_1\omega_2}{f_1f_2},$$

so that $a_1 + a_2$ is in $ASM(X)$ as well. Moreover, $A := \pi_*(\omega_1 \wedge \omega_2 / f_1 f_2)$ is an almost semi-meromorphic current that coincides with $a_1 \wedge a_2$ outside the set $\pi(\text{sing}(\pi) \cup V(f_1) \cup V(f_2))$. If we had two other representations of a_j we would get an almost semi-meromorphic A' that coincides generically with $a_1 \wedge a_2$ on X . Because of the SEP thus $A = A'$. Thus we can define $a_1 \wedge a_2$ as A . It is readily verified that

$$a_2 \wedge a_1 = (-1)^{\deg a_1 \deg a_2} a_1 \wedge a_2$$

as usual. Moreover,

$$a_1 \wedge (a_2 + a_3) = a_1 \wedge a_2 + a_1 \wedge a_3$$

so $ASM(X)$ is an algebra.

Example 4.4. The most basic example of an (almost) semi-meromorphic current is the (principal value current of) a meromorphic form. Let f be a meromorphic $(k, 0)$ -form on X , i.e., (locally) $f = g/h$ where h is a holomorphic function that does not vanish identically on any irreducible component of X and g is a holomorphic $(k, 0)$ -form. By definition $g/h = g'/h'$ if and only if $g'h - gh'$ vanishes outside a set of positive codimension. In that case

$$(4.2) \quad g\left[\frac{1}{h}\right] = g'\left[\frac{1}{h'}\right]$$

outside a set of positive codimension. By the dimension principle, thus (4.2) holds everywhere. Thus there is a well-defined almost semi-meromorphic current $[f]$ associated with f . Notice that $ZSS([f])$ is contained in the pole set of the meromorphic function f , so unless X is smooth it may have codimension larger than 1. In fact, by a result by Malgrange, [21], it follows that $ZSS([f])$ is equal to the pole set of f . \square

The following lemma is crucial.

Lemma 4.5. *If a is almost semi-meromorphic in X , then there is a representation (4.1) such that f is nonvanishing in $X' \setminus \pi^{-1}ZSS(a)$.*

Proof. Let $V = ZSS(a)$ and assume that we have a representation (4.1) and that X' is smooth. Let Z' be an irreducible component of $Z = Z(f)$ that is not fully contained in $\pi^{-1}V$. We will show that there is a representation $a = \pi_*(\tilde{\omega}/g)$, where $\tilde{\omega}$ is smooth and g is nonvanishing on $Z' \cap Z_{reg}$. Taking this for granted, the lemma then follows by a finite induction over the number of irreducible components of Z not fully contained in π^*a .

Since X' is smooth, Z' is a Cartier divisor, and therefore there is a section s of a line bundle $L' \rightarrow X'$ that defines Z' . Since Z' is irreducible, f vanishes to a fixed order r along Z' , and it follows that $f = f'g$ where $f' = s^r$ and g is holomorphic and nonvanishing on $Z' \cap Z_{reg}$. Outside $\text{sing}(\pi) \cup Z = \text{sing}(\pi) \cup Z \cup \pi^{-1}V$ we have that $\omega = f\pi^*a$ and hence

$$(4.3) \quad \omega = f\pi^*a = f'\pi^*a$$

there. By continuity it follows that (4.3) must hold in $X' \setminus \pi^{-1}V$ since both sides are smooth there.

We now claim that $\tilde{\omega} := \omega/f'$ is smooth in X' . This is a local statement in X' so given a point in X' we can choose local coordinates t in a neighborhood \mathcal{U} of that

point and consider each coefficient of the form ω with respect to these coordinates. Thus we may assume that ω is a function. Then still $\omega = f'\gamma$ where γ is smooth. For all multiindices α we thus have that

$$(4.4) \quad \frac{\partial^\alpha \omega}{\partial t^\alpha} \bar{\partial} \frac{1}{f'} = 0$$

in $\mathcal{U} \setminus \pi^{-1}V$, since $f'\bar{\partial}(1/f') = 0$. By assumption $Z' \cap \pi^{-1}V$ has positive codimension in Z' . By the dimension principle it follows that (4.4) holds in \mathcal{U} for all α . From [2, Theorem 1.2] we conclude that $\tilde{\omega}$ is smooth in \mathcal{U} . It follows that $\tilde{\omega}$ is smooth in X' . \square

Theorem 4.6. *Assume that $a \in \text{ASM}(X)$. For each $\tau \in \mathcal{PM}(X)$ there is a unique pseudomeromorphic current T in X that coincides with $a \wedge \tau$ in $X \setminus ZSS(a)$ and such that $\mathbf{1}_{ZSS(a)}T = 0$.*

If h is a holomorphic tuple such that $Z(h) = ZSS(a)$, then

$$(4.5) \quad T = \lim_{\epsilon \rightarrow 0} \chi(|h|^2 v / \epsilon) a \wedge \tau.$$

We will usually denote the extension T by $a \wedge \tau$ as well.

Proof. Let $V = ZSS(a)$. If such an extension T exists then $T = \mathbf{1}_{X \setminus V}T = \mathbf{1}_{X \setminus V}a \wedge \tau$ and so T is unique and moreover (4.5) holds in view of Lemma 3.1.

Conversely, if the limit in (4.5) exists as a pseudomeromorphic current T in X , then it must coincide with $a \wedge \tau$ in $X \setminus V$. In particular, $\chi(|h|^2 v / \epsilon)T = \chi(|h|^2 v / \epsilon)a \wedge \tau$ for each $\epsilon > 0$ and hence, taking limits and using Lemma 3.1, we get $\mathbf{1}_{X \setminus V}T = T$, i.e., $\mathbf{1}_{ZSS(a)}T = 0$. To prove the theorem it is thus enough to verify that the limit in (4.5) exists as a pseudomeromorphic current.

In view of Lemma 4.5 we may assume that a has the form (4.1), where $Z = Z(f)$ is contained in $\pi^{-1}V$ and $\omega/f = \pi^*a$ in $X' \setminus \pi^{-1}V$. Let $\chi_\epsilon = \chi(|h|^2 v / \epsilon)$, so that $\pi^*\chi_\epsilon = \chi(|\pi^*h|^2 / \epsilon)$. By Proposition 2.4 there is $\tau' \in \mathcal{PM}(X')$ such that $\pi_*\tau' = \tau$. Thus

$$\chi_\epsilon a \wedge \tau = \chi_\epsilon a \wedge \pi_*\tau' = \pi_*(\pi^*\chi_\epsilon \pi^*a \wedge \tau') = \pi_*(\pi^*\chi_\epsilon \frac{\omega}{f} \wedge \tau').$$

In view of Proposition 3.4 and Lemma 3.1,

$$\pi^*\chi_\epsilon \frac{\omega}{f} \wedge \tau' \rightarrow \mathbf{1}_{X' \setminus \pi^{-1}V} \frac{\omega}{f} \wedge \tau'$$

when $\epsilon \rightarrow 0$. In particular, the limit is a pseudomeromorphic current. Thus the limit in (4.5) exists and is pseudomeromorphic. \square

Notice that if a is almost semi-meromorphic, τ is pseudomeromorphic, and W is any analytic set, then

$$(4.6) \quad \mathbf{1}_W(a \wedge \tau) = a \wedge \mathbf{1}_W\tau.$$

In fact, the equality holds in the open set $X \setminus ZSS(a)$ by (3.2) since a is smooth there. On the other hand both sides vanish on $ZSS(a)$ since $\mathbf{1}_{ZSS(a)}\mathbf{1}_W(a \wedge \tau) = \mathbf{1}_W\mathbf{1}_{ZSS(a)}(a \wedge \tau) = 0$ in view of (1.2) and (3.2). As a corollary we therefore have

Corollary 4.7. *Let $Z \subset X$ be an analytic subset of pure dimension. If τ is in \mathcal{W}_Z , then $a \wedge \tau$ is in \mathcal{W}_Z as well.*

In fact, certainly $a \wedge \tau$ has support on Z if τ has. Moreover, if $V \subset Z$ has positive codimension, then $\mathbf{1}_V(a \wedge \tau) = a \wedge \mathbf{1}_V\tau = 0$ if $\mathbf{1}_V\tau = 0$.

Remark 4.8. Sometimes it is not necessary to cut out precisely $ZSS(a)$ in (4.5). Let h be a holomorphic function that is generically nonvanishing on Z and such that $Z(h) \supset ZSS(a)$. If τ is in \mathcal{W}_Z , then $\{h \neq 0\} \cap Z$ has positive codimension in Z and hence $\mathbf{1}_{\{h \neq 0\} \cap Z} \tau = \tau$. Thus $\mathbf{1}_{\{h \neq 0\}} a \wedge \tau = a \wedge \mathbf{1}_{\{h \neq 0\}} \tau = a \wedge \mathbf{1}_{\{h \neq 0\} \cap Z} \tau = a \wedge \tau$ and therefore

$$\lim \chi(|h|^2 v / \delta) a \wedge \tau = a \wedge \tau.$$

□

Corollary 4.9. *Assume that $a_1, a_2 \in ASM(X)$ and $\tau \in \mathcal{PM}^X$. Then*

$$(4.7) \quad a_1 \wedge a_2 \wedge \tau = (-1)^{\deg a_1 \deg a_2} a_2 \wedge a_1 \wedge \tau.$$

This follows since both sides of (4.7) coincide outside $ZSS(a_1) \cup ZSS(a_2)$ and vanish on $ZSS(a_1) \cup ZSS(a_2)$. Notice that $ZSS(a_1 \wedge a_2) \subset ZSS(a_1) \cup ZSS(a_2)$ but that the inclusion may be strict. Therefore it is *not* true in general that $a_1 \wedge a_2 \wedge \tau = (a_1 \wedge a_2) \wedge \tau$. Take for instance $a_1 = 1/z_1$, $a_2 = z_1/z_2$, $\tau = \bar{\partial}(1/z_1)$. Then $(a_1 a_2) \tau = (1/z_2) \bar{\partial}(1/z_1)$ but $a_1 a_2 \tau = 0$. For similar reasons it is not true in general that $(a_1 + a_2) \wedge \tau = a_1 \wedge \tau + a_2 \wedge \tau$. However, as long as τ is in \mathcal{W} this kind of problems do not occur, cf. Remark 4.8 with $Z = X$.

Proposition 4.10. *If $a \in ASM(X)$, then $\mathbf{1}_{X \setminus ZSS(a)} \bar{\partial} a$ is in $ASM(X)$ and ∂a is in $ASM(X)$.*

Proof. Let $V = ZSS(a)$, and assume that $a = \pi_*(\omega/f)$ and that $Z(f) \subset \pi^{-1}V$, cf. Lemma 4.5. Then

$$\bar{\partial} a = \pi_* \frac{\bar{\partial} \omega}{f} + \pi_* \bar{\partial} \frac{1}{f} \wedge \omega.$$

Now, by (3.3),

$$(4.8) \quad \mathbf{1}_{X \setminus V} \bar{\partial} a = \pi_* \left(\mathbf{1}_{\pi^{-1}(X \setminus V)} \frac{\bar{\partial} \omega}{f} \right) + \pi_* \left(\mathbf{1}_{\pi^{-1}(X \setminus V)} \bar{\partial} \frac{1}{f} \wedge \omega \right) = \pi_* \left(\frac{\bar{\partial} \omega}{f} \right);$$

thus $\mathbf{1}_{X \setminus V} \bar{\partial} a \in ASM(X)$. For the last equality we have used Proposition 3.4 and the fact that $\bar{\partial}(1/f)$ has support on $\pi^{-1}V$.

Now, let $D = D' + \bar{\partial}$ be a Chern connection on L . Then

$$\partial a = \pi_* \left(\partial \frac{\omega}{f} \right) = \pi_* \frac{f \cdot D' \omega - D' f \wedge \omega}{f^2}$$

which is in $ASM(X)$. □

If a is almost smooth, then both ∂a and $\bar{\partial} a$ are almost smooth. If a_1 and a_2 are almost smooth, then also $a_1 \wedge a_2$ is almost smooth.

4.1. Residue of almost semi-meromorphic currents. In view of Proposition 4.10, in general if $a \in ASM(X)$, then

$$\bar{\partial} a = b + r,$$

where $b = \mathbf{1}_{X \setminus ZSS(a)} \bar{\partial} a$ is almost semi-meromorphic and $r = \mathbf{1}_{ZSS(a)} \bar{\partial} a$ has support on $ZSS(a)$. We call r the *residue (current)* of a . If $a = \pi_*(\omega/f)$, then, in view of the proof of Proposition 4.10,

$$r = \pi_* \left(\bar{\partial} \frac{1}{f} \wedge \omega \right).$$

Note that the residue current $\bar{\partial}(1/f)$ is the residue of the principal value current $1/f$. Similarly, the residue currents introduced, e.g., in [23, 1, 9] can be considered as residues of certain almost semi-meromorphic currents, generalizing $1/f$.

Example 4.11. Let us describe the construction of residue currents in [1]. Let f be a holomorphic section of a Hermitian vector bundle $E \rightarrow X$ and let σ be the section over $X \setminus Z(f)$ of the dual bundle with minimal norm such that $f\sigma = 1$. There is a modification $\pi: X' \rightarrow X$ that is a biholomorphism $X' \setminus \pi^{-1}Z(f) \simeq X \setminus Z(f)$ such that

$$\pi^*\sigma = \frac{\alpha}{s},$$

where s is a holomorphic section of a line bundle $L \rightarrow X'$ with zero set contained in $\pi^{-1}Z(f)$ and α is a smooth section of $L \otimes \pi^*E$. Thus

$$\pi^*(\sigma \wedge (\bar{\partial}\sigma)^{k-1}) = \frac{\alpha \wedge (\bar{\partial}\alpha)^{k-1}}{s^k}$$

is a section of $\Lambda^k(\pi^*E \oplus T_{0,1}^*(X'))$ in $X' \setminus \pi^{-1}Z(f)$. It follows that

$$U_k := \sigma \wedge (\bar{\partial}\sigma)^{k-1}$$

has a (necessarily unique) extension to an almost semi-meromorphic section of $\Lambda^k(E \oplus T_{0,1}^*(X))$, as the push-forward of $\alpha \wedge (\bar{\partial}\alpha)^{k-1}/s^k$. Clearly $ZSS(U_k) = Z(f)$. Now the residue current R in [1] is the residue of the almost semi-meromorphic current $\sum_k U_k$. If E is trivial with trivial metric, the coefficients of R are the Bochner-Martinelli residue currents introduced in [23]. \square

If a is almost semi-meromorphic and τ is any pseudomeromorphic current, we can define

$$(4.9) \quad \bar{\partial}a \wedge \tau := \bar{\partial}(a \wedge \tau) - (-1)^{\deg a} a \wedge \bar{\partial}\tau,$$

i.e., so that the Leibniz rule holds. It is easily checked that

$$\bar{\partial}a \wedge \tau = \lim_{\delta \rightarrow 0} \bar{\partial}\chi(|h|^2 v/\delta) a \wedge \tau + \lim_{\delta \rightarrow 0} \chi(|h|^2 v/\delta) \bar{\partial}a \wedge \tau = r + \mathbf{1}_{X \setminus ZSS(a)} \bar{\partial}a \wedge \tau$$

if $Z(h) = ZSS(a)$; here r has support on $ZSS(a)$. In particular this gives a way of defining products of residue of almost semi-meromorphic currents. For example, the Coleff-Herrera product $\bar{\partial}[1/f_1] \wedge \cdots \wedge \bar{\partial}[1/f_p]$ can be defined by inductively applying (4.9), and in [3] the first author used these ideas to define products of more general residue currents. Notice that in general $a_1 \wedge \bar{\partial}a_2$ is *not* equal to $\pm \bar{\partial}a_2 \wedge a_1$. For instance, $(1/z)\bar{\partial}(1/z) = 0$, whereas $\bar{\partial}(1/z) \cdot (1/z) = \bar{\partial}(1/z^2)$. The Coleff-Herrera product is (anti-)commutative when the codimension of $\{f_1 = \cdots = f_p = 0\}$ is at least p .

Remark 4.12. There are other (weighted) approaches to products of residue currents, see, e.g., [22, 26], which coincide with the products above under suitable conditions. \square

5. ACTION OF HOLOMORPHIC DIFFERENTIAL OPERATORS AND HOLOMORPHIC VECTOR FIELDS

In this section we assume that X is smooth. For simplicity we even assume that X is an open subset of \mathbb{C}^n . Let z be a holomorphic global coordinate system and let $dz := dz_1 \wedge \cdots \wedge dz_n$.

Theorem 5.1. (i) Assume that

$$(5.1) \quad a = \sum_{|I|=p}^I a_I \wedge dz_I$$

is in $ASM(X)$. Then each a_I is in $ASM(X)$.

(ii) Let ξ be a holomorphic vector field in X and assume that $a \in ASM(X)$. Then the contraction $\xi \lrcorner a$ and the Lie derivative $L_\xi a$ are in $ASM(X)$.

In particular, $\partial a_I / \partial z_j$, where a_I are as in (5.1), are in $ASM(X)$ for all I, j . In particular, if a has bidegree $(0, *)$ and L is a holomorphic differential operator, then La is in $ASM(X)$ as well.

Proof. First assume that $p = n$ and that $\mu \wedge dz$ is in $ASM(X)$. Assume that $\mu \wedge dz = \pi_*(\omega/f)$. If f is a section of $L \rightarrow X'$, then ω must be a section of $L \otimes K_{X'}$. Now $g = \pi^* dz$ is a generically nonvanishing section of $K_{X'}$. Thus $\mu' := \pi_*(\omega/fg)$ is almost semi-meromorphic in X , and $\mu' \wedge dz = \pm \pi_*(g\omega/fg) = \pm \mu \wedge dz$. It follows that $\mu = \pm \mu'$, and thus μ is in $ASM(X)$.

Now fix a multiindex J and let J^c be the complementary index. Then

$$a \wedge dz_{J^c} = \pm a_J \wedge dz.$$

In view of the first part thus a_J is in $ASM(X)$. Thus (i) is proved.

It follows from (i) that

$$\frac{\partial}{\partial z_j} \lrcorner a$$

is in $ASM(X)$. Thus $\xi \lrcorner a$ is in $ASM(X)$. Now $L_\xi a = \xi \lrcorner (\partial a) + \partial(\xi \lrcorner a)$ is in $ASM(X)$ in view of Proposition 4.10. \square

Remark 5.2. Clearly part (i) of Theorem 5.1 follows from part (ii). One can prove (ii) directly for any smooth X and any holomorphic vector field ξ in the following way. Assume that $\pi: X' \rightarrow X$ and $a = \pi_*(\omega/f)$ as before. Clearly $\xi' = d(\pi^{-1})\xi$ is a holomorphic vector field in $X' \setminus \text{sing}(\pi)$ and one can verify that it has a meromorphic extension across $\text{sing}(\pi)$. In fact, if s are local coordinates in X' , and z local coordinates on X , then

$$\frac{\partial}{\partial s} = \frac{\partial z}{\partial s} \frac{\partial}{\partial z}$$

and hence we can express $\partial/\partial z$ as a meromorphic combination of $\partial/\partial s_j$, with denominator $h = \det(\partial z/\partial s)$. It follows that $\xi' = \xi''/h$, where h is a section of $K_{X'}$. Thus

$$\xi \lrcorner a = \pi_* \frac{\xi'' \lrcorner \omega}{hf}$$

outside $\pi(\text{sing}(\pi))$ and so it must hold in X . \square

6. SOME FURTHER PROPERTIES OF \mathcal{PM} AND \mathcal{W}_Z

Again, let X be a reduced analytic space of pure dimension and $Z \subset X$ a subvariety of pure dimension. We already know that ∂ maps \mathcal{PM} into itself. We also have

Lemma 6.1. *If $\mu \in \mathcal{W}_Z$, then $\partial \mu \in \mathcal{W}_Z$.*

For the proof we need the following result that is of interest in itself.

Lemma 6.2. *A pseudomeromorphic current μ in \mathcal{PM}_x is in $(\mathcal{W}_Z)_x$ if and only if it has a representation*

$$\mu = \sum_{\ell} \pi_* \tau_{\ell},$$

where $\text{supp } \tau_{\ell} \subset \pi^{-1}Z$ for all ℓ and there is no $V \subset Z$ of positive codimension and ℓ such that $\text{supp } \tau_{\ell} \subset \pi^{-1}V$.

Proof. Assume first that μ is in $(\mathcal{W}_Z)_x$ and let us begin with any representation (2.6). Recall from the proof of Lemma 3.3 that for each subvariety $W \subset Z$ either $\text{supp } \tau_{\ell} \subset \pi^{-1}W$ or $\mathbf{1}_{\pi^{-1}W} \tau_{\ell} = 0$. Since

$$\mu = \mathbf{1}_Z \mu = \sum_{\ell} \pi_* (\mathbf{1}_{\pi^{-1}Z} \tau_{\ell}) = \sum_{\text{supp } \tau_{\ell} \subset \pi^{-1}Z} \pi_* \tau_{\ell}$$

it follows that we can delete from (2.6) all τ_{ℓ} such that $\text{supp } \tau_{\ell}$ is not contained in $\pi^{-1}Z$. Now let $V \subset Z$ be a subvariety of positive codimension in Z . Then

$$0 = \mathbf{1}_V \mu = \sum_{\ell} \pi_* (\mathbf{1}_{\pi^{-1}V} \tau_{\ell}) = \sum_{\text{supp } \tau_{\ell} \subset \pi^{-1}V} \pi_* \tau_{\ell}.$$

Thus we can delete from (2.6) all τ_{ℓ} such that $\text{supp } \tau_{\ell} \subset \pi^{-1}V$. Proceeding in this way for all $V \subset Z$ of positive codimension we end up with a representation as in the lemma. The converse follows from the same considerations. \square

Proof of Lemma 6.1. Take a representation of μ as in Lemma 6.2. Then, by Remark 2.1,

$$\partial \mu = \sum_{\ell} \pi_* \partial \tau_{\ell}$$

is also such a representation. \square

Remark 6.3. Notice that if τ is an elementary pseudomeromorphic current in \mathbb{C}_t^n and t^{α} is a monomial, then there is an elementary current τ' such that $t^{\alpha} \tau' = \tau$. In fact, by induction it is enough to assume that the monomial is t_1 . If t_1 is a residue factor in τ , then we just raise the power of t_1 in that factor one unit. Otherwise we take $\tau' = (1/t_1) \tau$. \square

We shall now see that this observation holds in more generality.

Proposition 6.4. *Assume that $\mu \in \mathcal{PM}_x$ where $x \in X$. If $h \in \mathcal{O}_x$ is not identically zero on any irreducible component of X at x , then there is $\mu' \in \mathcal{PM}_x$ such that $h\mu' = \mu$. If $\mu \in \mathcal{W}$, then we can choose $\mu' \in \mathcal{W}$.*

Remark 6.5. By a partition of unity we can get a global such μ' if μ and h are global. Note that if μ has compact support in $\mathcal{U} \subset X$ we can choose μ' with compact support in \mathcal{U} . \square

Notice that the proposition is not true if h is anti-holomorphic. In fact, if $\bar{z}\mu' = 1$, then $(1/z)\mu'$ is equal to $1/|z|^2$ outside 0. Thus $\lim \chi(|z|^2/\delta)\mu'/z$ does not exist, and hence μ' cannot be pseudomeromorphic, by Proposition 3.4 and (3.5).

Proof of Proposition 6.4. We may assume that $\mu = \pi_* \tau$, where $\pi : \mathcal{U} \rightarrow X$ and τ is elementary of the form (2.4). By Hironaka's theorem we can find a modification $\nu :$

$\mathcal{U}' \rightarrow \mathcal{U}$, such that $\nu^* \pi^* h$ is a monomial and $\nu^* t_j$ are monomials (times nonvanishing functions) in \mathcal{U}' . It follows that $\tau = \nu_* \tau'$, where

$$\tau' := \nu^* \alpha \wedge \left[\frac{1}{\nu^* t_1^{m_1}} \right] \cdots \left[\frac{1}{\nu^* t_k^{m_k}} \right] \bar{\partial} \left[\frac{1}{\nu^* t_{k+1}^{m_{k+1}}} \right] \wedge \cdots \wedge \bar{\partial} \left[\frac{1}{\nu^* t_r^{m_r}} \right]$$

is a sum of elementary currents τ_ℓ . By Remark 6.3, there are elementary currents τ'_ℓ in \mathcal{U}' such that $\nu^* \pi^* h \tau'_\ell = \tau_\ell$. Let $\mu' = \pi_* \nu_* \sum_\ell \tau'_\ell$. Then

$$h\mu' = \pi_* \nu_* \sum_\ell \nu^* \pi^* h \tau'_\ell = \pi_* \nu_* \sum_\ell \tau_\ell = \mu.$$

Thus the proposition follows for $\mu \in \mathcal{PM}_x$.

If μ is in \mathcal{W} then we can take $\mu' = (1/h)\mu$. Then μ' is in \mathcal{W} by (4.6) and Proposition 3.4, and $h\mu' = h(1/h)\mu = \mathbf{1}_{\{h \neq 0\}}\mu = \mu$, cf. Remark 3.5. \square

We now consider the analog of Theorem 5.1 for \mathcal{PM} and \mathcal{W} .

Theorem 6.6. *Assume now that X is smooth at $x \in X$, and let z be local holomorphic coordinate system.*

(i) *If*

$$(6.1) \quad \mu = \sum_{|I|=p}^I \mu_I \wedge dz_I$$

is a germ in \mathcal{PM}_x , then each μ_I is in \mathcal{PM}_x . If in addition μ is in \mathcal{W}_Z then each μ_I is in \mathcal{W}_Z .

(ii) *If ξ is a germ of a holomorphic vector field, then $\xi \lrcorner \mu$ and $L_\xi \mu$ are in \mathcal{PM}_x . If μ is in $(\mathcal{W}_Z)_x$ then $\xi \lrcorner \mu$ and $L_\xi \mu$ are in $(\mathcal{W}_Z)_x$.*

Notice that (ii) is not true for an anti-holomorphic vector field. For example, $\tau = (\partial/\partial \bar{z}) \lrcorner \bar{\partial}(1/z)$ is a nonzero current of degree 0 with support at 0. In view of the dimension principle, it cannot be pseudomeromorphic.

Proof. We will first assume that μ has bidegree $(n, *)$ so that $\mu = \hat{\mu} \wedge dz$ and show that $\hat{\mu}$ is pseudomeromorphic. We may assume that $\mu = \pi_*(\tau \wedge ds)$ where $\pi : \mathcal{U} \rightarrow X$, s are local coordinates in $\mathcal{U} \subset \mathbb{C}^m$, and τ is elementary. Since π has generically surjective differential, we can write $s = (s', s'') = (s'_1, \dots, s'_n, s''_{n+1}, \dots, s''_m)$ so that $h := \det(\partial\pi/\partial s') = \det(\partial z/\partial s')$ is generically nonvanishing in \mathcal{U} . By Proposition 6.4 and Remark 6.5 there is a pseudomeromorphic τ' with compact support in \mathcal{U} such that $h\tau' = \tau$ in \mathcal{U} . Now

$$\hat{\mu} \wedge dz = \pi_*(\tau \wedge ds) = \pi_*(\tau' \wedge h ds' \wedge ds'') = \pi_*(\tau' \wedge \pi^* dz \wedge ds'') = \pm \pi_*(\tau' \wedge ds'') \wedge dz.$$

Thus $\hat{\mu} = \pm \pi_*(\tau' \wedge ds'')$ is pseudomeromorphic.

In general, $\mu_I \wedge dz = \pm \mu \wedge dz_{I^c}$, where I^c is the complementary multiindex of I . It follows from above that μ_I is pseudomeromorphic. If in addition μ is in \mathcal{W}_Z then certainly each μ_I must have support on Z as well. Moreover, if $V \subset Z$ has positive codimension, then

$$0 = \mathbf{1}_V \mu = \sum_{|I|=p}^I (\mathbf{1}_V \mu_I) \wedge dz_I,$$

in view of (3.2), and hence $\mathbf{1}_V \mu_I = 0$ for each I . Thus (i) follows.

The first statement of (ii) follows immediately from (i). Since $L_\xi \mu = \partial(\xi \lrcorner \mu) + \xi \lrcorner (\partial \mu)$, the second statement follows, in view of Lemma 6.1. \square

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DEPARTMENT OF MATHEMATICAL SCIENCES, DIVISION OF MATHEMATICS, UNIVERSITY OF GOTHENBURG AND CHALMERS UNIVERSITY OF TECHNOLOGY, SE-412 96 GÖTEBORG, SWEDEN

E-mail address: matsa@chalmers.se, wulcan@chalmers.se